PHYSICS NYB-10/11 Winter 2007

Lecture 13: Continuous charge distributions: electric potential

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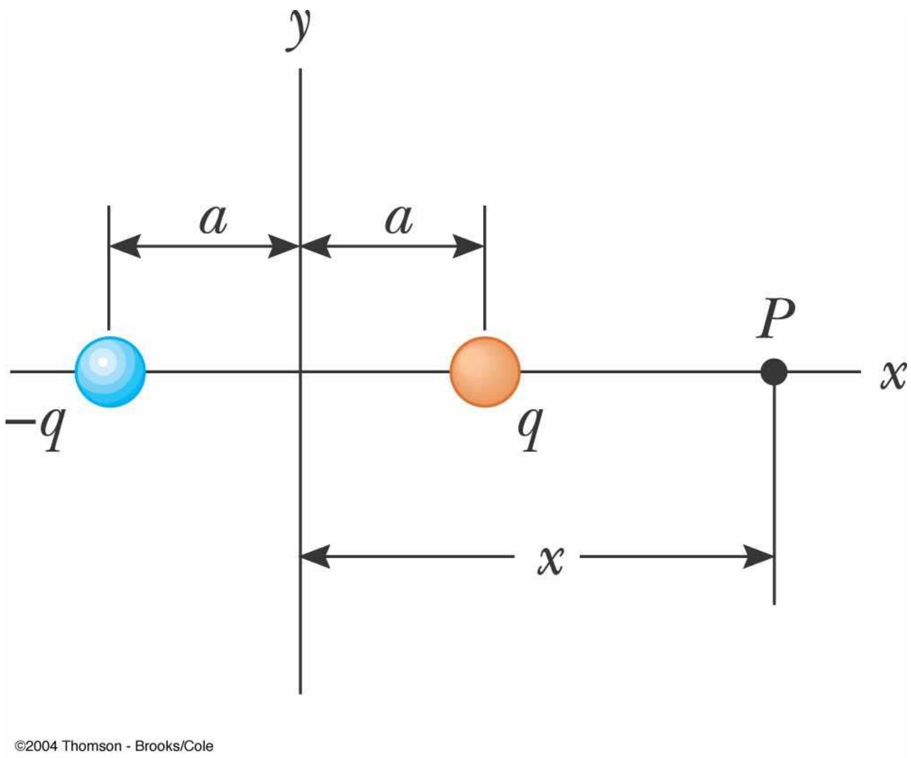
Review

- So far, we've seen how point charges exert forces $\vec{F}_e$ on each other.
- We've seen that this is due to the fact that a point charge has an electric field $\vec{E}$, and this electric field exerts a force on other charged particles placed inside it.
- We've also seen that a point charge has an electric potential V associated with it, and when another charge is placed inside a potential, there is potential energy U_e present.

Review

- Then we saw how this all comes together inside conducting wires connected to batteries to lead to flow of charge called current, I.
- The charges in a current can do work when they pass through a resistance R.
- Everything we've done so far concerned *point* charges only.

Review



Find the potential at point P from this electric dipole. Find the potential energy that will be added to the system if a proton is placed at point P. Find the electric field at point P. Find the force on a proton placed at point P.

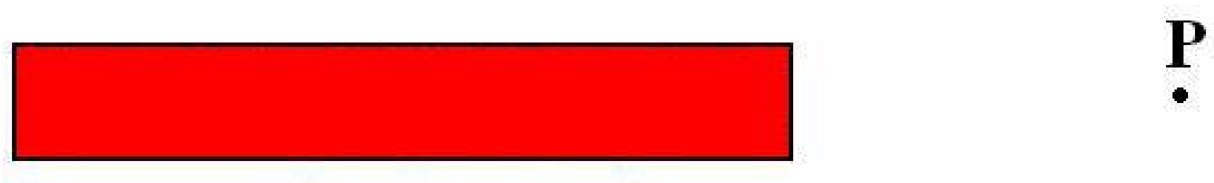
Review

To find the potential from two different charges, we add the potentials from each charge. So $V = k_e \dfrac{q}{x-a} - k_e \dfrac{q}{x+a}$. The potential energy from placing a proton at point P will be $U = eV$ where V is given by the previous expression.

The electric field is also the sum of the fields created by the two charges, but now we have to keep in mind the fact that the field is a vector pointing away from positive charges and towards negative charges. So $\vec{E} = k_e \dfrac{q}{(x-a)^2}\hat{i} - k_e \dfrac{q}{(x+a)^2}\hat{i}$. The force on a proton placed in this field will be $\vec{F} = e\vec{E}$.

Continuous charge distributions

Remember the very first lecture of the term. I rubbed a plastic rod with some fur, which led to the rod (and fur) picking up (opposite) net charge. Using what we currently know, would you be able to tell me the what the potential V and field $\vec{E}$ was near the tip of the rod?

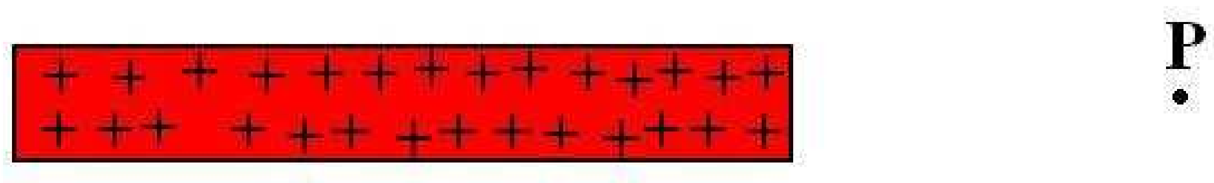


Continuous charge distributions

Well, yes and no... We only know how to find the potential or field for point charges, and a plastic rod is clearly not a point.

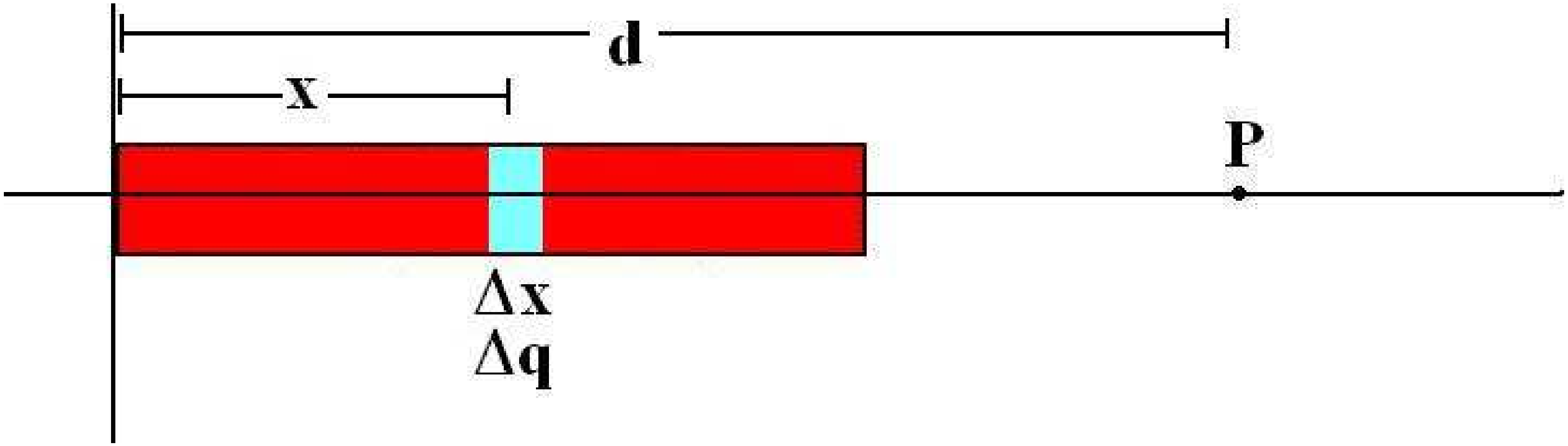
However, a plastic rod *is made of a whole bunch of points*.

We *do* know how to find the potential and field for each of these points. And we *do* know how to add...



Continuous charge distributions

So let's look at the potential from a small part of the rod, of length Δx and carrying an amount of charge Δq. If we take a small enough piece, it is perfectly ok to treat it as a point.



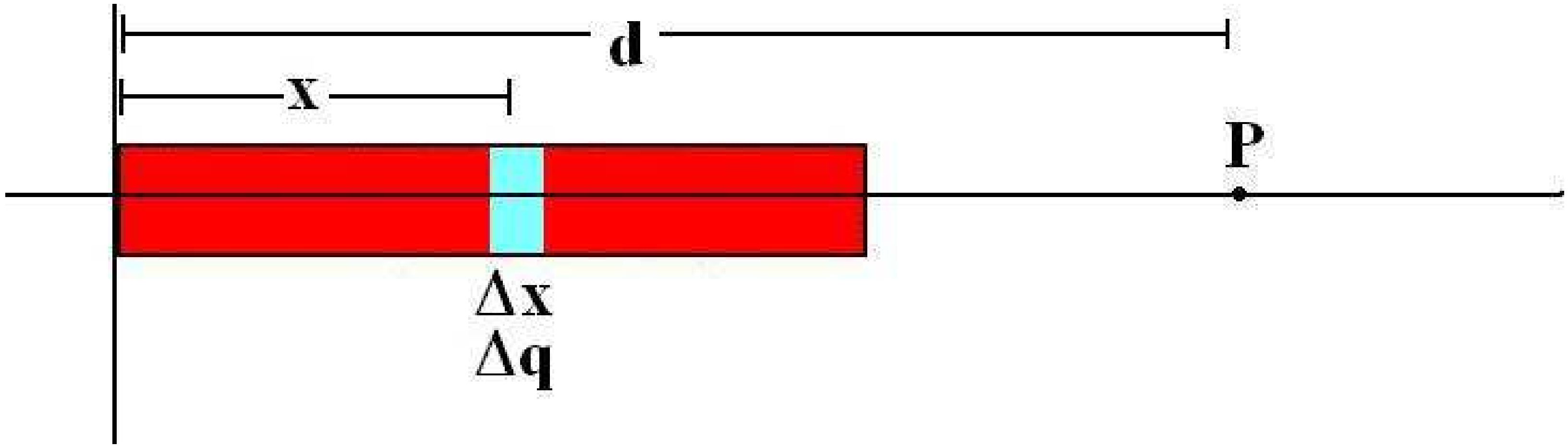
The small potential due to this small piece of the rod is

$$\Delta V = k_e \frac{\Delta q}{r} .$$

But how much charge is on the piece? What is Δq equal to?

Continuous charge distributions

Let's imagine that the *total* charge on the rod is Q. Imagine that this charge is spread out uniformly over the entire rod. Then we have Q Coulombs spread out over a rod of length L. This means there are Q/L Coulombs per meter on the rod. We'll call this last value the *linear charge density*, $\lambda \equiv Q/L$. Since we are thinking of a piece of the rod that is Δx meters long, it will carry a charge of $\Delta q = \lambda \Delta x$.



$$\Delta V = k_e \frac{\lambda \Delta x}{r}$$

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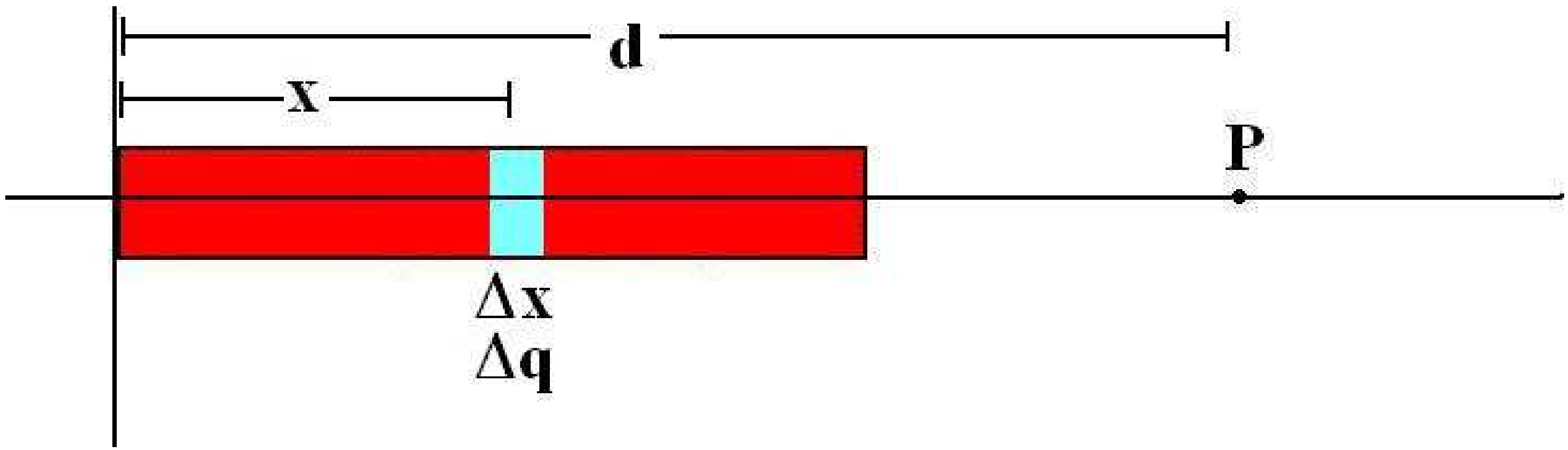
Continuous charge distributions

To get the total potential due to the entire rod, we need to add the potential due to every small piece. So

$$V_{tot} = \Delta V_1 + \Delta V_2 + ... \equiv \sum_i \Delta V_i.$$

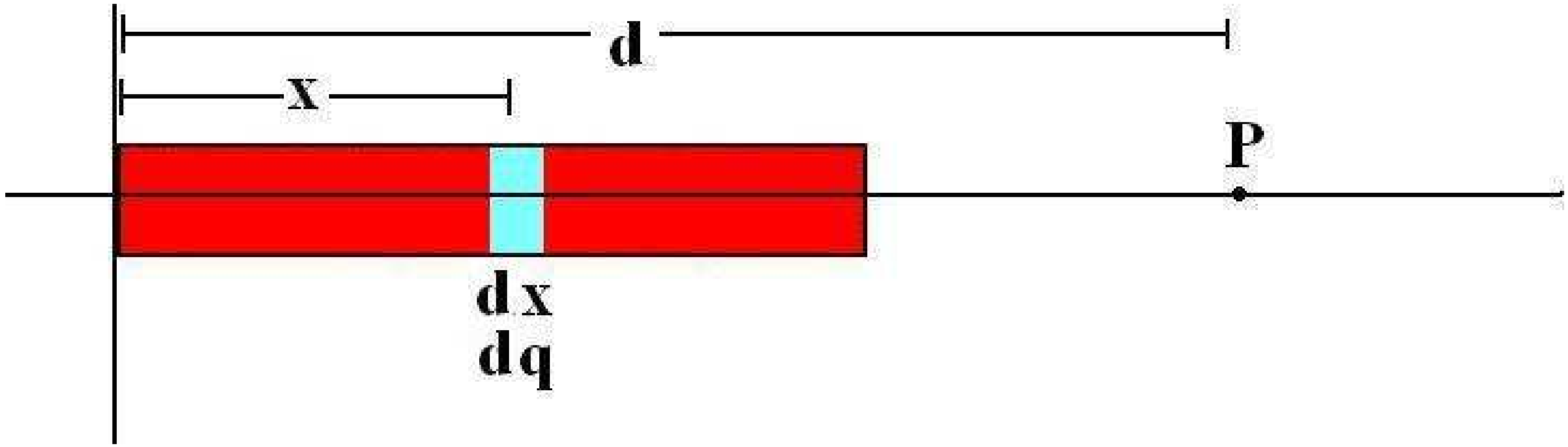
Note however that the distance r from the point P to each small piece is different. We can write $r_i = d - x_i$



$$V_{tot} = \sum_i \Delta V_i = \sum_i k_e \frac{\lambda \Delta x_i}{d - x_i}.$$

Continuous charge distributions

The last step is to make each piece infinitely small, so that $\Delta x_i \to dx$ and the sum becomes an infinite sum of infinitely small pieces, in other words *an integral*.



$$V_{tot} = \sum_i \Delta V_i = \sum_i k_e \frac{\lambda \Delta x_i}{d - x_i} \to V_{tot} = \int dV = \int_0^L k_e \frac{\lambda dx}{d - x} \quad .$$

Continuous charge distributions

Well, actually the *last* last step is to evaluate the integral.

$$V_{tot} = \int_0^L k_e \lambda \frac{dx}{d-x} = k_e \lambda \int_0^L \frac{dx}{d-x}$$

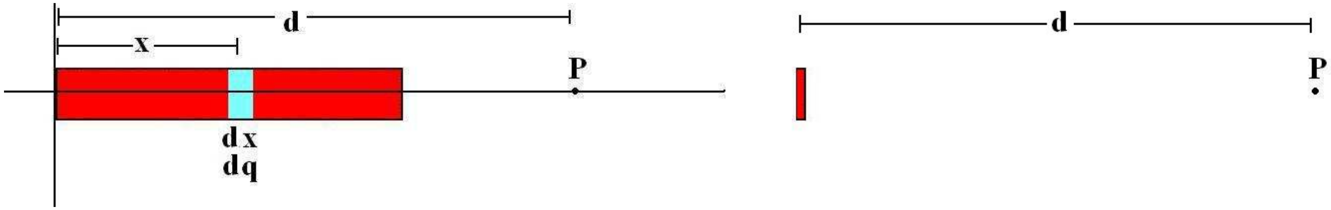
First we'll make a change of variables, $d - x = u$ which means $dx = -du$ and the boundaries of integration become from d to $d - L$. So

$$\begin{aligned}
V_{tot} &= k_e \lambda \int_d^{d-L} \frac{-du}{u} \\
&= -k_e \lambda \left[\ln(u)\right]_d^{d-L} \\
&= -k_e \lambda \left[\ln(d-L) - \ln(d)\right] \\
&= k_e \lambda \ln\left(\frac{d}{d-L}\right) = k_e \frac{Q}{L} \ln\left(\frac{d}{d-L}\right)
\end{aligned}$$

Continuous charge distributions

$$V_{tot} = k_e \frac{Q}{L} \ln\left(\frac{d}{d-L}\right)$$

Let's check if this result makes sense, at least in some simplified situation. Imagine the rod was very very short, then we should get back the same result as for a point charge.



Using what you learned in math, you can check that $\lim_{L\to 0} \frac{1}{L} \ln(d/(d-L)) = \frac{1}{d}$, which means that if the rod is short we get $V = k_e \frac{Q}{d}$, exactly as we'd expect for a point charge.

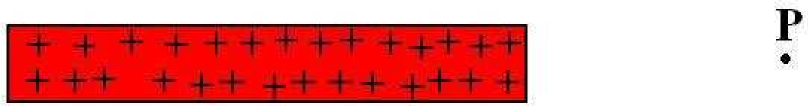
Continuous charge distributions

- When faced with a continuous charge distribution, use the following steps:
  - Choose your coordinates wisely.
  - Check if you can use *symmetry arguments* to simplify the problem.
  - Choose one small piece of the distribution containing a charge Δq.
  - Write down the potential at the point P for this single piece.

Continuous charge distributions

- Note that so far, you haven't done anything harder than finding the potential for a single point charge.
  - The key step comes next in writing Δq in terms of the charge *density* and the size of the small piece.
  - Make sure everything is written in terms of constants and the coordinates you chose.
  - Now take a sum of all the small pieces the distribution is made of by taking an integral; *make sure you correctly identify which coordinate(s) to integrate, and over what range.*
  - It's always a good idea to check that your answer reduces to something you know in an appropriate limit.

Linear charge density



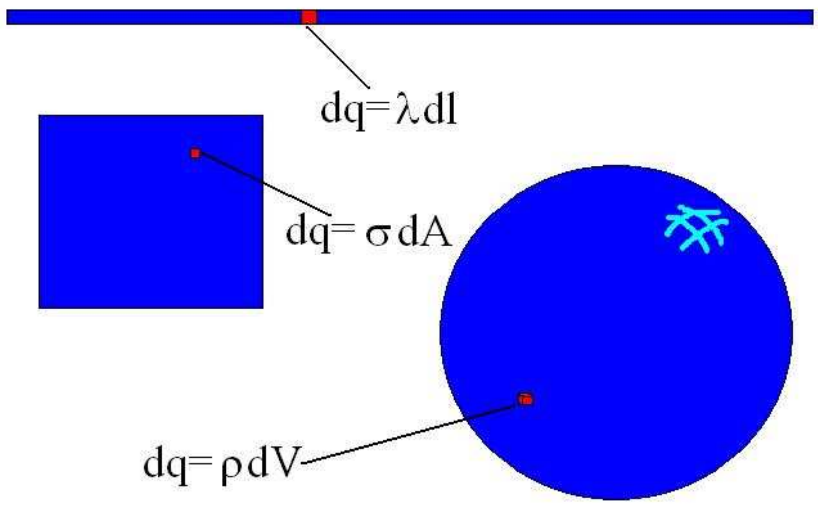
A charged rod has a length of 100 cm and a charge of 10 nC placed evenly over it. How much charge does a 10 cm piece of this rod have?

This is easy... If there are 10 nC on a 100 cm long piece, there is 1 nC on a piece one tenth as long. Basically, we can say there are 10 nC/m, and multiply this by 10 cm. We'll say that 10 nC/m is the *linear charge density* of the rod. We'll denote this as λ.

Surface and volume charge density

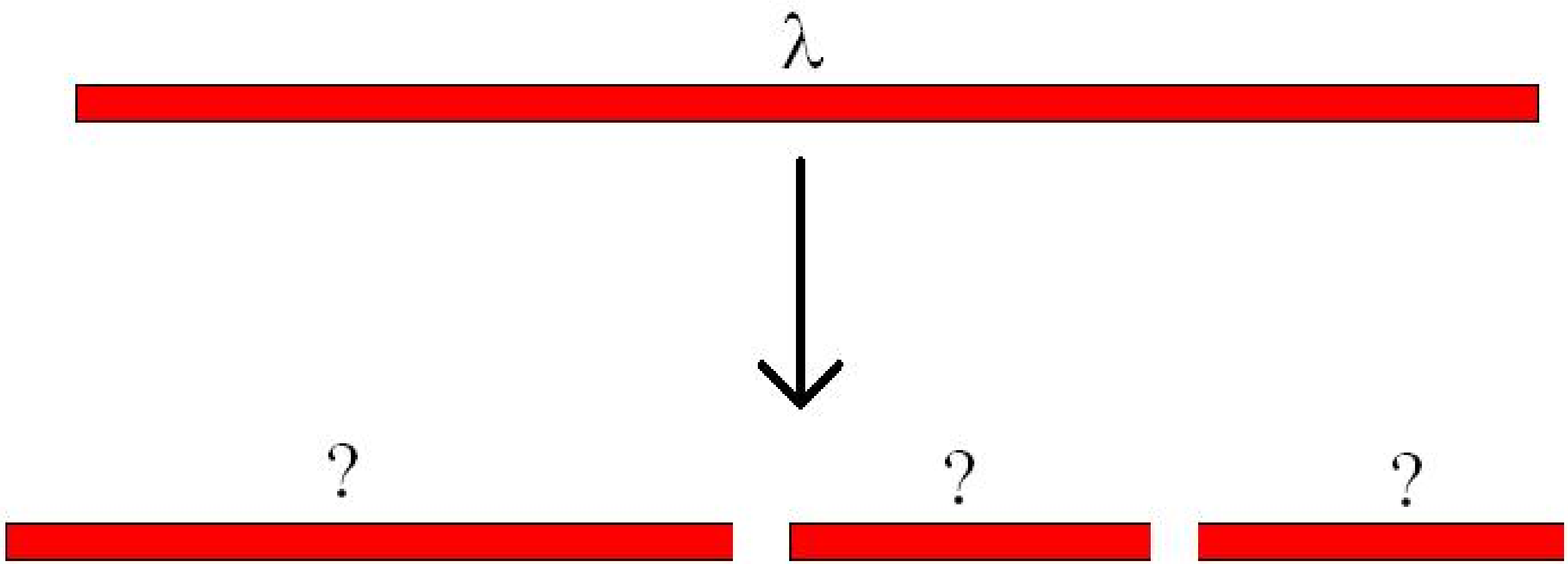
When the charge is distributed over a surface, we'll be interested in the amount of charge per meter squared. We call this the *surface charge density*, σ.

When the charge is distributed over a volume, we'll be interested in the charge per meter cubed. We call this the *volume charge density*, ρ.



Examples

A rod of uniform charge density λ and length l is cut up into three pieces of length $l/2$, $l/4$ and $l/4$. What is the total charge on each piece, and what is the linear charge density on each piece?



Examples

There are λ Coulombs per meter on the rod, so the total charge is λl C. On the piece of length $l/2$, there will be $\lambda l/2$ C, while the other two pieces have a charge of $\lambda l/4$ C.

Since the charge density is uniform, all three pieces will have a charge density of exactly λ C/m.

Examples

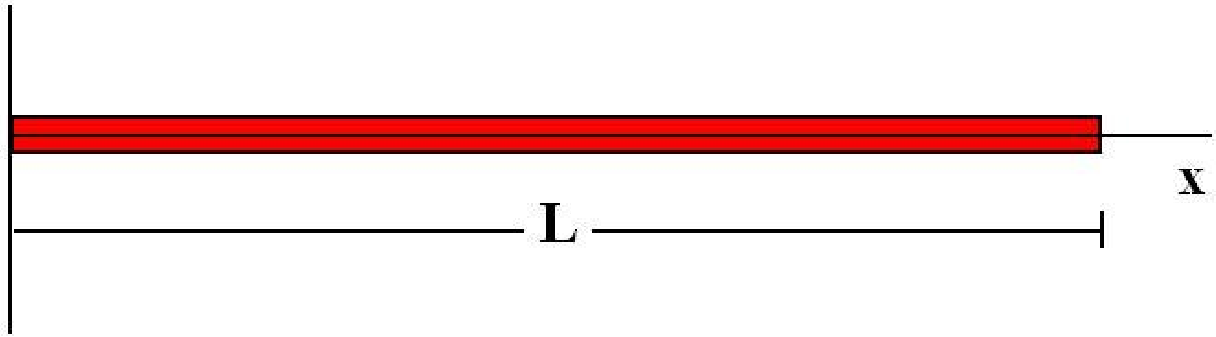
An insulating sphere with a diameter of 5 cm has a uniform charge density of $\rho = 4\ \mu$C/m^3. What is its total charge?

If the same charge was placed on a conductor of the same size, what would be the surface charge density?

The total charge is $Q = \rho V = \rho \frac{4\pi r^3}{3}$. Using the given value for ρ and the fact that the radius of the spere is 2.5 cm, we find a total charge of $Q = 0.26$ nC. If this charge was placed on a conducting sphere of the same radius, it would spread evenly over the surface of the sphere, and the surface density would be $\sigma = \frac{Q}{A} = \frac{Q}{4\pi r^2} = 33.3$ nC/m^2.

Examples

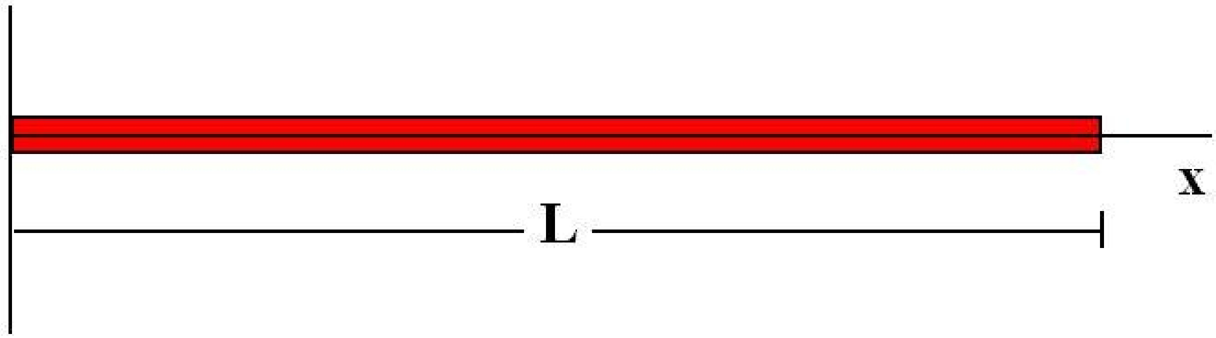
What is the total charge on this rod, where $\lambda = (3x - 4)$ μC/m and $L = 20$ cm?



Now we have to be careful, because the linear density is not the same everywhere, so we can't simply say that $Q = \lambda L$. Instead, every small piece of length Δx on the rod has a charge $\Delta q = \lambda(x)\Delta x$ where $\lambda(x)$ is the density at the location of the piece we are talking about. If we look at infinitely small pieces of the rod, and add up all their infinitely small amounts of charge, we get an integral

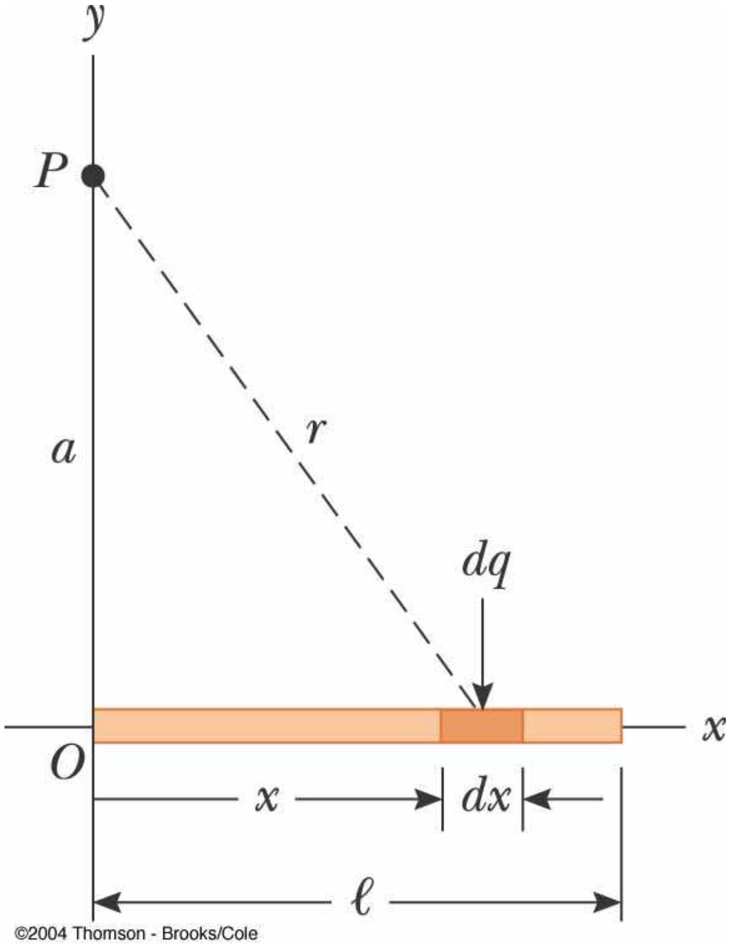
Examples

What is the total charge on this rod, where $\lambda = (3x - 4)$ μC/m and $L = 20$ cm?



$$
\begin{aligned}
Q &= \int dq = \int_0^L \lambda(x)dx = \int_0^L (3x-4)dx \\
&= \left[\frac{3}{2}x^2 - 4x\right]_0^L = \left(\frac{3}{2}L^2 - 4L\right)\ \mu\text{C} \\
&= \frac{3}{2}(0.2)^2 - 4(0.2) = -0.74\ \mu\text{C}
\end{aligned}
$$

Examples



If this rod has a uniform linear charge density λ, find the potential at point P.

Examples

The potential due to the small piece of length dx we've chosen is $dV = k_e \frac{dq}{r}$, where $dq = \lambda dx = \frac{Q}{l} dx$ and $r = \sqrt{a^2 + x^2}$. So $dV = k_e \frac{\lambda dx}{\sqrt{a^2 + x^2}}$. Now we need to take the sum of the potential due to every piece of the rod, going from $x = 0$ to $x = l$. So we need to perform the following integral

$$\begin{aligned} V = \int dV &= \int_0^l k_e \frac{\lambda dx}{\sqrt{x^2 + a^2}} \\ &= \lambda k_e \left[\ln \left(x + \sqrt{x^2 + a^2} \right) \right]_0^l \\ &= \lambda k_e \left[\ln \left(l + \sqrt{l^2 + a^2} \right) - \ln \left(a \right) \right] \end{aligned}$$

Examples

$$V = k_e \frac{Q}{l} \left[\ln\left(l + \sqrt{l^2 + a^2}\right) - \ln\left(a\right) \right]$$

Again, we can check that if the rod were very short, we would get the result for a point charge by checking that $\lim_{l \to 0} \frac{1}{l} \left[\ln\left(l + \sqrt{l^2 + a^2}\right) - \ln\left(a\right) \right] = \frac{1}{a}$ so that

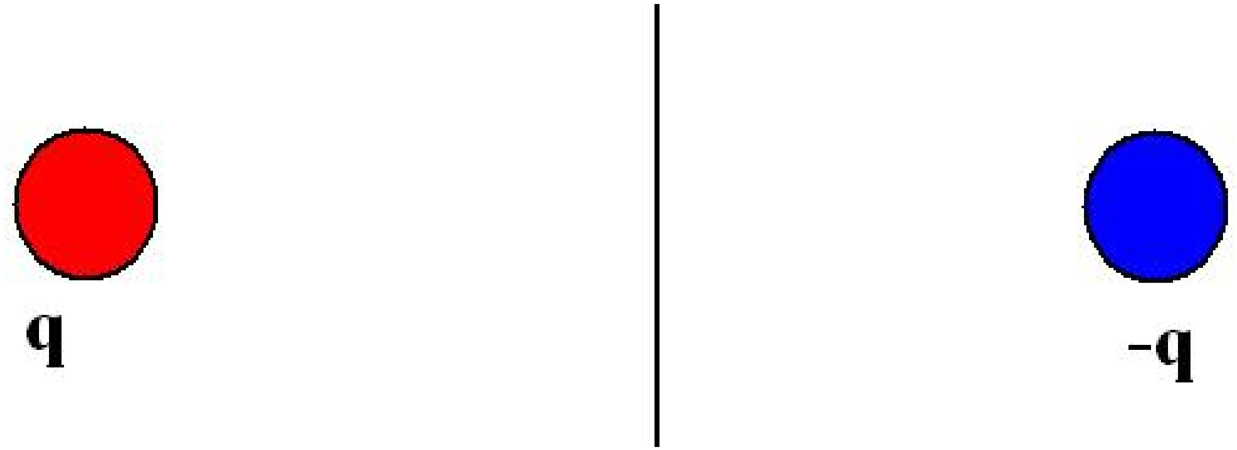
$$V = k_e \frac{Q}{a}$$

as it should.

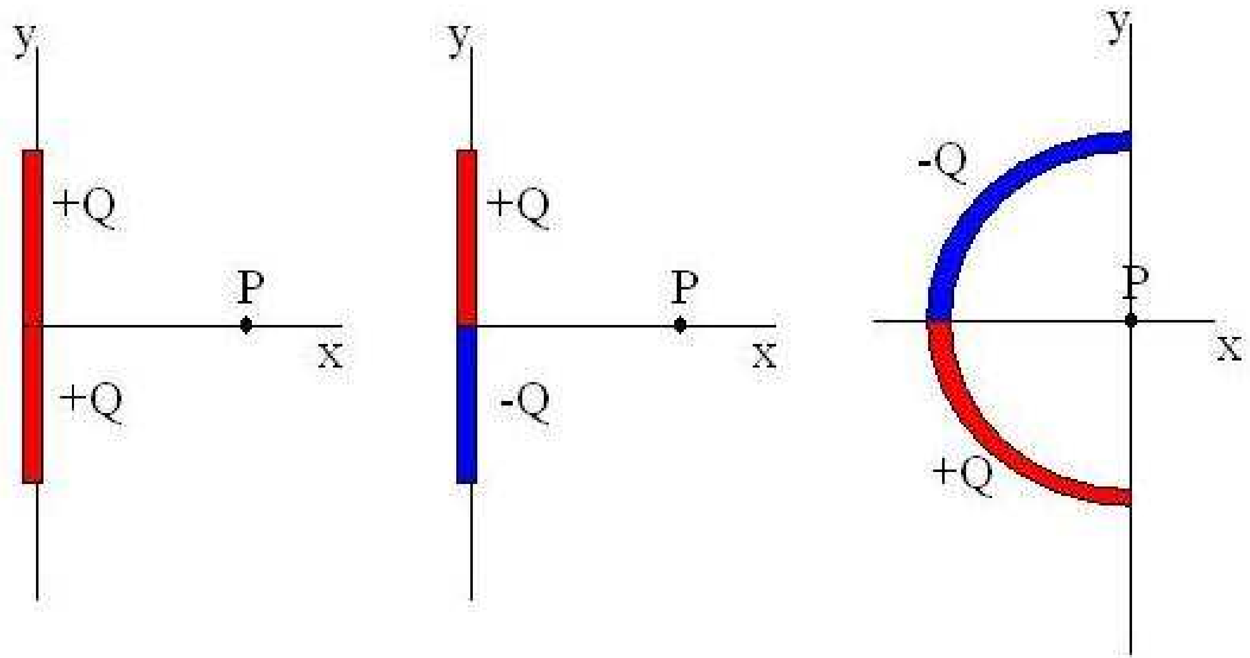
Using symmetry

Many times, you are able to gain very important information about the potential or the electric field simply by looking at the way the source charges are arranged.

For example, in the example below, you should be able to tell that the potential is zero on the line shown without using any math...

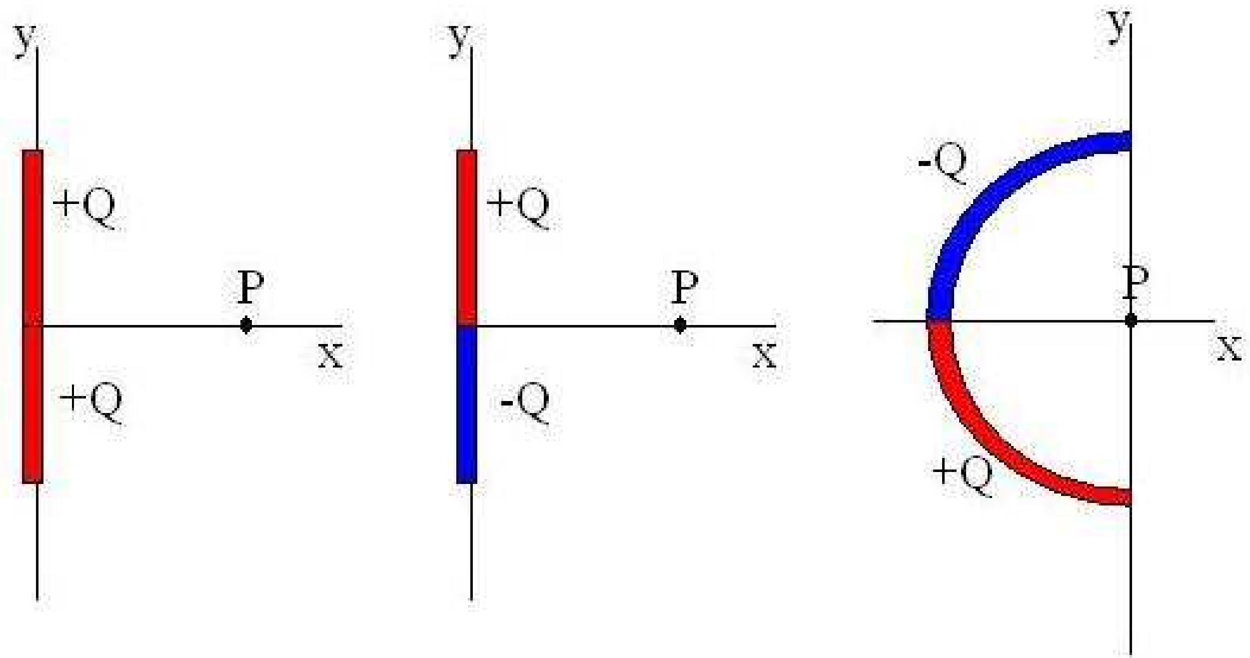


Examples



Question: The figure above shows three non conducting rods. Each has a uniform charge Q along the top and bottom halves. For each rod, what is the electric potential at point P?

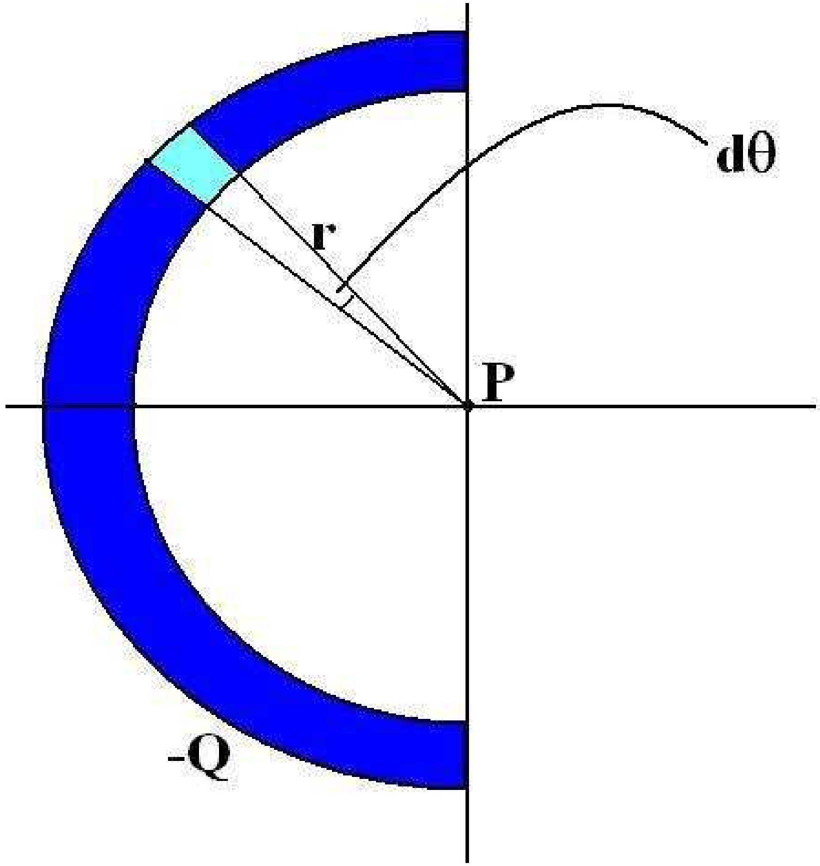
Examples



The first situation is simply twice the one we just found in the last example. So the potential will be $V = 2k_e \frac{Q}{l}\left[\ln\left(l + \sqrt{l^2 + a^2}\right) - \ln\left(a\right)\right]$ where l is half the length of the rod and a is the perpendicular distance to P. The other two cases have equal and opposite charge distributions exactly canceling the potential at P.

Examples

What is the potential at point P at the center of a semi-circular arc of uniformly distributed charge $-Q$ and radius r?



Examples

The small piece pictured in the diagram has a charge dq and therefore creates a potential $dV = k_e \frac{dq}{r}$ at point P. It has a length $ds = r d\theta$, and since there is a total charge $-Q$ distributed over a total length πr on the semi-circle, we know that $\lambda = -\frac{Q}{\pi r}$. This means $dq = \lambda ds = \lambda r d\theta = -\frac{Q}{\pi r} r d\theta$.

So now we can find the total potential

$V = -k_e \int_0^\pi \frac{Q d\theta}{\pi r} = -k_e \frac{Q}{r}$.

What to read for next lecture

- 23.5, 25.5